Assignment 4. Solutions.

Problems. March 15.

1. Find the curvature of the curve

$$\mathbf{r}(t) = t^4\ \mathbf{i} + \ln t\ \mathbf{j} + t\ \mathbf{k}$$

at the point where $t = 1$.

Solution.

We will use the formula

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = 4t^3\ \mathbf{i} + \frac{1}{t}\ \mathbf{j} + \mathbf{k},$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = 12t^2\ \mathbf{i} - \frac{1}{t^2}\ \mathbf{j}.$$

Then

$$\mathbf{v}(1) = 4\ \mathbf{i} + \mathbf{j} + \mathbf{k},$$

$$\mathbf{a}(1) = 12\ \mathbf{i} - \mathbf{j}.$$

$$\mathbf{v(1)} \times \mathbf{a(1)} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 1 \\ 12 & -1 & 0 \end{vmatrix} = \mathbf{i} + 12\mathbf{j} - 16\mathbf{k}.$$

$$|\mathbf{v(1)} \times \mathbf{a(1)}| = \sqrt{1 + 12^2 + 16^2} = \sqrt{401}.$$

$$|\mathbf{v}(1)| = \sqrt{4^2 + 1 + 1} = \sqrt{18} = 3\sqrt{2}.$$

Therefore,

$$\kappa(1) = \frac{\sqrt{401}}{(3\sqrt{2})^3} = \frac{\sqrt{401}}{54\sqrt{2}}.$$

2. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ for the curve

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + \ln \cos t\ \mathbf{k}$$

at the point where $t = 0$.

Solution.

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j} - \tan t\ \mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = \sec t = \frac{1}{\cos t}.$$

Note that there is no absolute value above since $\cos t > 0$ if t is close to 0.

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = -\sin t \cos t\ \mathbf{i} + \cos^2 t\ \mathbf{j} - \sin t\ \mathbf{k}$$

So,

$$\mathbf{T}(0) = \mathbf{j}.$$

Now we compute $\mathbf{N}$.

$$\frac{d}{dt}\mathbf{T}(t) = (-\cos^2 t + \sin^2 t)\mathbf{i} - 2\cos t \sin t\ \mathbf{j} - \cos t\ \mathbf{k} = -\cos 2t\ \mathbf{i} - \sin 2t\ \mathbf{j} - \cos t\ \mathbf{k}.$$

Then

$$\frac{d}{dt}\mathbf{T}(0) = -\ \mathbf{i} -\ \mathbf{k},$$

and

$$\left|\frac{d}{dt}\mathbf{T}(0)\right| = \sqrt{1+1} = \sqrt{2}.$$

Therefore,

$$\mathbf{N}(0) = \frac{\frac{d}{dt}\mathbf{T}(0)}{\left|\frac{d}{dt}\mathbf{T}(0)\right|} = -\frac{1}{\sqrt{2}}\ \mathbf{i} - \frac{1}{\sqrt{2}}\ \mathbf{k}.$$

Now we find $\mathbf{B}$ by the formula $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Therefore,

$$\mathbf{B}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{\sqrt{2}}\ \mathbf{i} + \frac{1}{\sqrt{2}}\ \mathbf{k}.$$

Problems. March 17.

1. $f(x, y) = y/x^2$. (a) Find the function's domain, (b) find the function's range, (c) describe the function's level curves, (d) find the boundary of the function's domain, (e) determine if the domain is an open region, a closed region, or neither, (f) decide if the domain is bounded or unbounded.

   Solution.

   (a) The domain consists of those point (x, y), where $x \neq 0$. So the domain is the entire plane without the y-axis.

   (b) If we fix x and vary y, we see that the function can take on any values. So, the range is $\mathbb{R}$ (all real numbers).

   (c) To find the level curves, consider the equation

   $$y/x^2 = c$$

or

$$y = cx^2.$$

These are parabolas (when $c \neq 0$). But since $x \neq 0$, these parabolas do not contain their vertex (which is at the origin). If $c = 0$, the curve is the line $y = 0$ without the point $(0, 0)$.

(d) The boundary of the domain is the y-axis.

(e) The domain is open.

(f) The domain is unbounded.

2. $f(x, y) = \sqrt{9 - x^2 - y^2}$. (a) Find the function's domain, (b) find the function's range, (c) describe the function's level curves, (d) find the boundary of the function's domain, (e) determine if the domain is an open region, a closed region, or neither, (f) decide if the domain is bounded or unbounded.

Solution.

(a) The domain consists of those point (x, y), where $x^2 + y^2 \leq 9$. This is a disk of radius 3 centered at the origin (together with the boundary circle).

(b) We see that $f(x, y) \geq 0$ and that $f(x, y) \leq \sqrt{9} = 3$. So, the range is the interval $[0, 3]$.

(c) To find the level curves, consider the equation

$$\sqrt{9 - x^2 - y^2} = c$$

or

$$x^2 + y^2 = c^2 - 9.$$

These are circles centered at the origin (when $0 \leq c < 3$). When $c = 3$, we get a single point $(0, 0)$.

(d) The boundary of the domain is the circle $x^2 + y^2 = 9$.

(e) The domain is closed.

(f) The domain is bounded.

3. $f(x, y) = \tan^{-1}(y/x)$. (a) Find the function's domain, (b) find the function's range, (c) describe the function's level curves, (d) find the boundary of the function's domain, (e) determine if the domain is an open region, a closed region, or neither, (f) decide if the domain is bounded or unbounded.

Solution.

(a) The domain consists of those point (x, y), where $x \neq 0$. So the domain is the entire plane without the y-axis.

(b) If we fix x and vary y, we see that y/x can take on any values. Therefore, $\tan^{-1}(y/x)$ can take on any values between $-\pi/2$ and $\pi/2$. So, the range is the interval $(-\pi/2, \pi/2)$.

(c) To find the level curves, consider the equation

$$\tan^{-1}(y/x) = c$$

or

$$y = x \tan c.$$

These are straight lines through the origin. But since $x \neq 0$, these lines do not contain the origin.

(d) The boundary of the domain is the y-axis.

(e) The domain is open.

(f) The domain is unbounded.

4. Sketch a typical level surface for the function

$$f(x, y, z) = \ln(x^2 + y^2 + z^2).$$

Solution.

Level surfaces are given by $\ln(x^2 + y^2 + z^2) = c$, that is

$$x^2 + y^2 + z^2 = e^c.$$

These are spheres centered at the origin.

Problems. March 19.

1. Find the limit

$$\lim_{(x,y)\to(0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$$

Solution.

By direct substitution

$$\lim_{(x,y)\to(0,0)} \cos \frac{x^2 + y^3}{x + y + 1} = \cos \frac{0}{1} = 1.$$

2. Find the limit

$$\lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$$

Solution.

$$\lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1} = \lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{(\sqrt{x} - \sqrt{y+1})(\sqrt{x} + \sqrt{y+1})}$$

$$= \lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{1}{\sqrt{x} + \sqrt{y+1}} = \frac{1}{\sqrt{4} + \sqrt{3+1}} = \frac{1}{4}.$$

3. At what points (x, y) in the plane are the functions continuous?

   (a) $g(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$,

   (b) $g(x, y) = \dfrac{1}{x^2 - y}$.

   Solution.

   (a) g is continuous wherever it's defined, that is the entire plane without the points where $x^2 - 3x + 2 = 0$. This gives $x = 1$ and $x = 2$. So, the function is continuous in the entire plane without the lines $x = 1$ and $x = 2$.

   (b) g is continuous wherever it's defined, that is the entire plane without the points where $x^2 - y = 0$. This is a parabola. So, the function is continuous in the entire plane without the parabola $y = x^2$.

4. Show that the function

$$f(x, y) = \frac{xy}{|xy|}$$

   has no limit as $(x, y) \to (0, 0)$.

   Solution.

   Consider the line $y = x$. Along this line

$$f(x, x) = \frac{xx}{|xx|} = \frac{x^2}{x^2} = 1.$$

   Consider the line $y = -x$. Along this line

$$f(x, -x) = \frac{-xx}{|-xx|} = \frac{-x^2}{x^2} = -1.$$

   Using the two-path test we see that the limit does not exist.

5. Find the limit of

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

   as $(x, y) \to (0, 0)$ or show that the limit does not exist.

   Solution.

   Consider the line $y = 0$. Along this line

$$f(x, 0) = \frac{x^2}{x^2} = 1.$$

   Consider the line $x = 0$. Along this line

$$f(0, y) = \frac{-y^2}{y^2} = -1.$$

   Using the two-path test we see that the limit does not exist.

Problems. March 22.

1. Find f_x, f_y, and f_z if $f(x,y,z) = yz\ln(xy)$.

   Solution.

   First let us write the function as follows $f(x,y,z) = yz(\ln x + \ln y)$. Then
$$f_x = \frac{yz}{x}, \qquad f_y = z\ln(xy) + z, \qquad f_z = y\ln(xy).$$

2. Find all the second partial derivatives of the function
$$h(x,y) = xe^y + y + 1.$$

   Solution.
$$h_x = e^y, \qquad h_y = xe^y + 1$$
$$h_{xx} = 0, \qquad h_{xy} = h_{yx} = e^y, \qquad h_{yy} = xe^y.$$

3. Find the value of $\partial x/\partial z$ at the point $(1,-1,-3)$ if the equation
$$xz + y\ln x - x^2 + 4 = 0$$
   defines x as a function of the two independent variables y and z.

   Solution.

   Differentiating $xz + y\ln x - x^2 + 4 = 0$ implicitly with respect to z we get
$$\frac{\partial x}{\partial z}\,z + x + \frac{y}{x}\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0,$$
$$\frac{\partial x}{\partial z}(z + \frac{y}{x} - 2x) = -x,$$
$$\frac{\partial x}{\partial z} = -\frac{x}{z + \frac{y}{x} - 2x}.$$
   The value at the given point $(1,-1,-3)$ is
$$\frac{\partial x}{\partial z} = -\frac{1}{-3-1-2} = \frac{1}{6}.$$

4. Show that the function $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$ satisfies the Laplace equation $f_{xx} + f_{yy} + f_{zz} = 0$.

   Solution.
$$f_x = -6xz, \qquad f_{xx} = -6z,$$
$$f_y = -6yz, \qquad f_{yy} = -6z,$$
$$f_z = 6z^2 - 3(x^2+y^2), \qquad f_{zz} = 12z.$$
   Therefore,
$$f_{xx} + f_{yy} + f_{zz} = -6z - 6z + 12z = 0.$$

5. Show that the function $w = \cos(2x + 2ct)$ is a solution of the wave equation

$$w_{tt} = c^2 w_{xx}.$$

Solution.

$$\begin{aligned} w_t &= -2c\sin(2x+2ct), & w_{tt} &= -4c^2\cos(2x+2ct), \\ w_x &= -2\sin(2x+2ct), & w_{xx} &= -4\cos(2x+2ct). \end{aligned}$$

$$w_{tt} = -4c^2\cos(2x+2ct) = c^2(-4\cos(2x+2ct)) = c^2 w_{xx}.$$

Problems. March 24.

1. (a) Express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating with respect to t.

   (b) Evaluate dw/dt at the given value of t.

$$w = z - \sin xy, \qquad x = t, \qquad y = \ln t, \qquad z = e^{t-1}; \qquad t = 1.$$

Solution.

(a) Using the Chain Rule:

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (-y\cos xy)1 + (-x\cos xy)\frac{1}{t} + 1e^{t-1} \\ &= -\ln t\cos(t\ln t) - \cos(t\ln t) + e^{t-1}. \end{aligned}$$

If we first express w in terms of t, we get

$$w = e^{t-1} - \sin(t\ln t).$$

Then

$$\frac{dw}{dt} = e^{t-1} - \cos(t\ln t)(\ln t + 1),$$

which agrees with the previous computation.

(b)

$$\left.\frac{dw}{dt}\right|_{t=1} = 1 - \cos 0 = 0.$$

2. (a) Express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating.

(b) Evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

$$w = \ln(x^2+y^2+z^2), \qquad x = ue^v \sin u, \qquad y = ue^v \cos u, \qquad z = ue^v;$$
$$(u, v) = (-2, 0).$$

Solution.

(a) Using the Chain Rule:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$= \frac{2x}{x^2+y^2+z^2}(e^v \sin u + ue^v \cos u) + \frac{2y}{x^2+y^2+z^2}(e^v \cos u - ue^v \sin u) + \frac{2z}{x^2+y^2+z^2}e^v$$
$$= \frac{2ue^v \sin u(e^v \sin u + ue^v \cos u) + 2ue^v \cos u(e^v \cos u - ue^v \sin u) + 2ue^v e^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}$$
$$= \frac{2ue^{2v}\sin^2 u + 2ue^{2v}\cos^2 u + 2ue^{2v}}{2u^2e^{2v}} = \frac{4ue^{2v}}{2u^2e^{2v}} = \frac{2}{u}.$$

Similarly,

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$
$$= \frac{2x}{x^2+y^2+z^2}ue^v \sin u + \frac{2y}{x^2+y^2+z^2}ue^v \cos u + \frac{2z}{x^2+y^2+z^2}ue^v$$
$$= \frac{2u^2e^{2v}\sin^2 u + 2u^2e^{2v}\cos^2 u + 2u^2e^{2v}}{2u^2e^{2v}} = \frac{4u^2e^{2v}}{2u^2e^{2v}} = 2.$$

If we express w directly in terms of u and v, we get

$$w = \ln(x^2+y^2+z^2) = \ln(u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v})$$
$$= \ln(2u^2e^{2v}) = \ln 2 + 2\ln|u| + 2v.$$

From here we see that

$$\frac{\partial w}{\partial u} = \frac{2}{u}, \quad \text{and} \quad \frac{\partial w}{\partial v} = 2.$$

These agree with the computations above.

(b)

$$\left.\frac{\partial w}{\partial u}\right|_{(-2,0)} = -1,$$
$$\left.\frac{\partial w}{\partial v}\right|_{(-2,0)} = 2.$$

3. Write the Chain Rule for $\partial w/\partial p$ if $w = f(x,y,z,v)$, $x = g(p,q)$, $y = h(p,q)$, $z = j(p,q)$, $v = k(p,q)$.

Solution.

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$

4. Find $\partial z/\partial u$ when $u = 0$, $v = 1$ if $z = \sin xy + x \sin y$, $x = u^2 + v^2$, $y = uv$.

Solution.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$= (y\cos xy + \sin y)2u + (x\cos xy + x\cos y)v.$$

Since $x(0,1) = 1$, $y(0,1) = 0$, we have

$$\left.\frac{\partial z}{\partial u}\right|_{(0,1)} = 2.$$

Problems. March 26.

1. The equation $xy + y^2 - 3x - 3 = 0$ defines y as a function of x. Use Theorem 8 from Section 14.4 to find the value of dy/dx at the point $(-1,1)$.

Solution.

Denote $F(x,y) = xy + y^2 - 3x - 3$. Then by Theorem 8 from Section 14.4 we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}.$$

At the given point $(-1,1)$:

$$\frac{dy}{dx} = 2.$$

2. Find the gradient of the function $g(x,y) = x^2/2 - y^2/2$ at the point $(\sqrt{2},1)$. Then sketch the gradient together with the level curve that passes through the point.

Solution.

$$\nabla g = g_x\mathbf{i} + g_y\mathbf{j} = x\mathbf{i} - y\mathbf{j}.$$
$$\left.\nabla g\right|_{(\sqrt{2},1)} = \sqrt{2}\mathbf{i} - \mathbf{j}.$$

The level curves are hyperbolas

$$\frac{x^2}{2} - \frac{y^2}{2} = c.$$

To find c, put $x = \sqrt{2}$, $y = 1$ into the latter equation. Then $c = 1/2$. So, the level curve passing through the given point is the parabola $x^2 - y^2 = 1$.

After you have sketched the parabola, draw the gradient vector at the given point. The gradient should be orthogonal to the level curve.

3. Find the derivative of the function $f(x, y) = 2x^2 + y^2$ at the point $P_0(-1, 1)$ in the direction of the vector $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$.

Solution.

Since $\mathbf{A}$ is not a unit vector, we have to normalize it. Denote

$$\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = 4x\mathbf{i} + 2y\mathbf{j}.$$

$$\nabla f\Big|_{(-1,1)} = -4\mathbf{i} + 2\mathbf{j}.$$

Therefore,

$$(D_{\mathbf{u}}f)\Big|_{(-1,1)} = \nabla f\Big|_{(-1,1)} \cdot \mathbf{u} = -\frac{12}{5} - \frac{8}{5} = -4.$$

4. Find the directions in which the function $f(x, y) = x^2y + e^{xy}\sin y$ increases and decreases most rapidly at $P_0(1, 0)$. Then find the derivatives of the function in these directions.

Solution.

The function f increases most rapidly in the direction of ∇f and f decreases most rapidly in the direction of $-\nabla f$.

We have

$$\nabla f = (2xy + ye^{xy}\sin y)\mathbf{i} + (x^2 + xe^{xy}\sin y + e^{xy}\cos y)\mathbf{j}.$$

$$\nabla f\Big|_{(1,0)} = 2\mathbf{j}.$$

Therefore, at the given point the function increases most rapidly in the direction of $\mathbf{j}$ and decreases most rapidly in the direction of $\mathbf{i}$.

The derivatives in these directions are correspondingly $|\nabla f|\Big|_{(1,0)} = 2$ and $-|\nabla f|\Big|_{(1,0)} = -2$.

5. In what directions is the derivative of the function $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?

Solution.

The derivative is equal to zero in the directions perpendicular to the gradient.

$$\nabla f = \frac{2x(x^2 + y^2) - 2x(x^2 - y^2)}{(x^2 + y^2)^2}\,\mathbf{i} + \frac{-2y(x^2 + y^2) - 2y(x^2 - y^2)}{(x^2 + y^2)^2}\,\mathbf{j}$$

$$= \frac{4xy^2}{(x^2 + y^2)^2}\,\mathbf{i} + \frac{-4yx^2}{(x^2 + y^2)^2}\,\mathbf{j}.$$

$$\nabla f\Big|_{(1,1)} = \mathbf{i} - \mathbf{j}.$$

Vectors that are perpendicular to the latter one are $\mathbf{i} + \mathbf{j}$ and $-\mathbf{i} - \mathbf{j}$. Normalizing these vectors, we get that the derivative of f at the point P is equal to zero in the directions

$$\frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} \qquad \text{and} \qquad -\frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{j}.$$

Problems. March 29.

1. Write an equation for the tangent line to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.

Solution.

Let us denote $f(x, y) = x^2 - xy + y^2$. Then the equation of the tangent line is

$$f_x(-1, 2)(x + 1) + f_y(-1, 2)(y - 2) = 0.$$

$$f_x = 2x - y, \qquad f_x(-1, 2) = -4,$$

$$f_y = -x + 2y, \qquad f_y(-1, 2) = 5.$$

Therefore, the tangent line is

$$-4(x + 1) + 5(y - 2) = 0,$$

$$4x - 5y + 14 = 0.$$

2. Find equations for (a) the tangent plane (b) the tangent line at the point $P_0(1, -1, 3)$ on the surface $x^2 + 2xy - y^2 + z^2 = 7$.

Solution.

Let $f(x, y, z) = x^2 + 2xy - y^2 + z^2$. Then

$$\nabla f = (2x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j} + 2z\mathbf{k}.$$

$$\nabla f\Big|_{P_0} = 4\mathbf{j} + 6\mathbf{k}.$$

(a) The tangent plane is given by

$$f_x(1,-1,3)(x-1) + f_y(1,-1,3)(y+1) + f_z(1,-1,3)(z-3) = 0,$$

$$4(y+1) + 6(z-3) = 0,$$

$$4y + 6z - 14 = 0.$$

(b) The normal line is parallel to the gradient and passing through the point P_0. Therefore, it has the equations

$$\begin{cases} x = 1, \\ y = -1 + 4t, \\ z = 3 + 6t. \end{cases}$$

3. By about how much will

$$f(x,y,z) = e^x \cos yz$$

change as the point $P(x,y,z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

Solution.

Taking the unit vector in the direction of the vector given above, we get

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}.$$

Now compute the gradient of f.

$$\nabla f = e^x \cos yz\mathbf{i} - ze^x \sin yz\mathbf{j} - ye^x \sin yz\mathbf{k}.$$

Since P_0 is the origin, we have

$$\nabla f\Big|_{P_0} = \mathbf{i}.$$

Then the required change approximately equals

$$(\nabla f\Big|_{P_0} \cdot \mathbf{u})ds = \frac{1}{\sqrt{3}}0.1 = \frac{1}{10\sqrt{3}}.$$

4. Find the linearization of the function $f(x,y) = (x+y+2)^2$ at the points (a) (0,0), (b) (2,2).

Solution.

The linearization at the point (a,b) is given by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

$$f_x = 2(x+y+2), \qquad f_y = 2(x+y+2).$$
$$f_x(0,0) = 4, \qquad f_y(0,0) = 4.$$
$$f_x(2,2) = 12, \qquad f_y(2,2) = 12.$$

(a) The linearization at $(0,0)$ is

$$L(x,y) = 4 + 4x + 4y.$$

(a) The linearization at $(2,2)$ is

$$L(x,y) = 36 + 12(x-2) + 12(y-2) = -12 + 12x + 12y.$$

5. Find the linearization $L(x,y)$ of the function $f(x,y) = x^2/2 + xy + y^2/4 + 3x - 3y + 4$ at $P_0(2,2)$. Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x,y) \approx L(x,y)$ over the rectangle $R : |x-2| \le 0.1, |y-2| \le 0.1$.

Solution.

$$f_x = x + y + 3, \qquad f_y = x + y/2 - 3$$
$$f_x(2,2) = 7, \qquad f_y(2,2) = 0$$
$$f(2,2) = 11.$$

So,

$$L(x,y) = 11 + 7(x-2) + 0(y-2) = 7x - 3.$$

Now estimate the error of the approximation.

$$f_{xx} = 1, \qquad f_{xy} = f_{yx} = 1, \qquad f_{yy} = 1/2.$$

Let $M = 1$ be the largest of these numbers. Then

$$|E(x,y)| \le \frac{1}{2}M(|x-2| + |y-2|)^2 \le \frac{1}{2}(0.1+0.1)^2 = 0.02.$$